PHYSICAL REVIEW E VOLUME 53, NUMBER 4 APRIL 1996

Stochastic process for the dynamics of the turbulent cascade

Ricardo Lima
Centre de Physique Théorique CNRS, Luminy Case 907, F13288 Marseille Cedex 9, France

R. Vilela Mendes
Grupo de Física-Matemática, Complexo II, Universidade de Lisboa, Avenida Gama Pinto 2, P-1699 Lisboa Codex, Portugal
(Received 17 January 1995)

The data on velocity increments over a distance l and turbulent energy dissipation on a box of size l are, in the inertial range, well described by the phenomenological multifractal models of fully developed turbulence. These quantities and models, however, do not specify the time correlations and therefore are not a complete parametrization for the dynamics of the turbulent cascade. An asymmetric stochastic process on a tree is proposed as a phenomenological model to parametrize the dynamics of the turbulent cascade. In the framework of this model, any concrete assumptions about the physical parameters of the cascade are easily related to the structure of the temporal correlations and, therefore, may be checked by comparison with experiment. As an example, we show how to relate the decay of the temporal correlations to the scale dependence of the lifetimes of the eddies at different length scales. From the hierarchical structure of the proposed process and the coexistence of many different time scales, we predict specific deviations from exponential behavior which may be checked by two-point correlation experiments.

PACS number(s): 47.27.Gs, 05.40.+j

I. INTRODUCTION

One of the most interesting phenomena in fully developed turbulence is the occurrence of an energy cascade from the macroscopic length scale L of the experimental apparatus down to smaller and smaller length scales. Length scales in the range $L \gg l \gg \eta$, η being the scale where the fragmentation process is stopped by dissipation, are said to be in the *inertial range.* In the inertial range, viscosity effects are not important and Kolmogorov [1] proposed long ago a scaling theory with conserved energy transfer between length scales. From scale invariance and the assumption that turbulence is space filling, it would follow that the velocity fluctuation $\delta v(l)$ over an active eddy of size l scales as

$$\langle \|\delta v(l)\|^p \rangle \sim l^{\zeta_p} \tag{1}$$

with $\zeta_p = p/3$. A related quantity is the locally averaged energy dissipation ϵ_l over a ball of diameter l,

$$\langle \epsilon_l^p \rangle \sim l^{\tau_p} . \tag{2}$$

Kolmogorov's refined scaling hypothesis [2] providing a relation between the exponents.

There is now strong evidence [3–6] that ζ_p is not $p/3$. This is traced back to the fact that turbulence is not space filling [7] and the volume of the active eddies may change when the energy is transferred from the scale l_n to the scale l_{n+1}. This leads naturally to a fractal structure for the cascade with fractal dimension less than 3. For example in the simple β model [8] the rate of energy transfer

$$E_n \sim \frac{\delta v_n^3}{l_n}$$

does not change along the cascade but the total mass of the active eddies is multiplied by M at each step. Then the exponent ζ_p in Eq. (1) becomes

$$\zeta_p = ph + 3 - D$$

with $h = (D-2)/3$, where D is the fractal dimension, related to M by

$$\log_2 M = D - 3$$

if the length scales are related by $l_n = l_0 2^{-n}$.

The β model as well as a *log-normal model* [2] for the distribution of E_n are, however, in contradiction with the experimental results on moments of higher order for the velocity structure functions [5,6]. This fact led to the proposal of several multifractal models [9–13]. Here one assumes that, at each scale l_n, there are several distinct multipliers $M_n(k)$ which are chosen according to some probability law. That is, the energy transfer may take place according to several distinct dimensional routes. Requiring a fixed energy transfer rate one obtains

$$\frac{\delta v_n^3(k)}{l_n} = M_{n+1}(k) \frac{\delta v_{n+1}^3(k)}{l_{n+1}} . \tag{3}$$

Hence at scale l_n the velocity fluctuation in each eddy depends on the fragmentation history which is defined by the product $M_1 M_2 \cdots M_n$. Then

$$\delta v_n \sim l_n^{1/3} \left[\prod_{i=1}^{n} M_i \right]^{-1/3} \tag{4}$$

and

$$\langle \|\delta v_n(l_n)\|^p \rangle \sim l_n^{p/3} \int \textstyle\prod_{i=1}^n dM_i M_i^{1-p/3} P(M_1 \cdots M_n) , \tag{5}$$

$P(M_1 \cdots M_n)$ being the occurrence probability for the sequence $M_1 \cdots M_n$. At the level of precision of the existing experiments, agreement with the data is already obtained if one assumes independent fragmentations

$$P(M_1 \cdots M_n) = \prod_{i=1}^{n} P(M_i) \tag{6}$$

and a simple binomial process

$$P(M) = C\delta_1 + (1-C)\delta_{1/2} \ . \tag{7}$$

Two important ingredients in this and in most other multifractal models are the existence of the same multiplier probability density at all levels of the cascade and the statistical independence of the multipliers at one level from those at previous levels. Although statistical independence is contradicted by experiment [14], the uncorrelated level-independent multiplier models give a good representation of the experimental results [14,15]. Ultimately these models should be understood in more fundamental terms from solutions of the Navier-Stokes equation. The idea that the energy transfer down the cascade takes place according to several distinct dimensional routes receives some support from direct numerical simulations [16] of the Navier-Stokes equation, which show that localized structures are largely responsible for intermittency effects and deviations from Kolmogorov scaling. But, on the other hand, recent approximate numerical solutions [17] of the Navier-Stokes equation indicate extremely small scaling corrections in the inertial range. This might mean that

(1) in future experiments at higher Reynolds number the scaling corrections will eventually go away, or that

(2) closer attention should be paid, in the interpretation of the experimental results, to the narrowing of the inertial range for higher order moments, or that

(3) a much higher density of large wave number vectors is needed, in the numerical solutions, to capture the intermittency effects.

Whatever the situation may be, it points to the need for further refined experiments in this field. It would also be important to obtain accurate information not only on the space correlations

$$\langle \|v(x+l)-v(x)\|^p \rangle = \langle \|\delta v(l)\|^p \rangle \tag{8}$$

but on the time correlations

$$\langle \|\delta v(l,t+dt)\delta v(l,t)\|^q \rangle \tag{9}$$

as well. The space correlations (8) reflect the geometry and the scale dependence of the energy transfer down the cascade, whereas the decay of the temporal correlations (9) is a direct test of the decay dynamics of the eddies at scales $\geq l$.

Assumptions of the type of Eqs. (6) and (7) only define the probability distribution at each level of the cascade tree (Fig. 1). They make no statement concerning the time evolution and the time scales of the eddies in the cascade. The multifractal models are phenomenological models which are essentially static, in the sense that they only describe the time-averaged statistical properties of

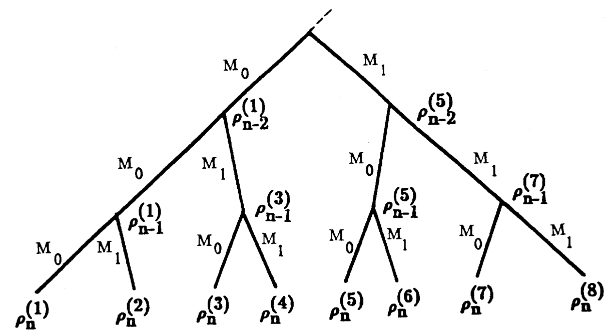

FIG. 1. State space at level n for a dyadic turbulent cascade.

the cascade. Even at the phenomenological level, one has to go beyond the usual multifractal models to be able to characterize the time correlations. That is, a definite dynamics must be specified for the stochastic process. That is the purpose of the present paper. By specifying a stochastic process for the dynamics of the cascade tree one should be able, at least, to parametrize the time correlations in terms of a small number of parameters to be fixed by experiment. On the other hand some of the predictions, namely, the deviations from exponential decay of the time correlations, are, we believe, largely independent of the precise value of the parameters.

The physical quantities for which we model the time correlations are quantities like the velocity fluctuations $\delta v(l)$ or the energy dissipation ϵ_l. We will be concerned with the internal dynamics of the cascade itself [18], not with the changes induced by the fluid motion. Therefore our model will apply, for example, to quantities like

$$\langle \epsilon_l(x(t),t)\epsilon_l(x(0),0) \rangle ,$$

$x(t)$ being the Lagrangian position of the fluid element originally at $x(0)$.

The construction of the stochastic process and the parametrization of time correlations from the solutions of the Chapman-Kolomogorov equation are discussed in the next section.

II. A STOCHASTIC PROCESS FOR THE CASCADE AND THE PARAMETRIZATION OF TIME CORRELATIONS

For definiteness we will consider a binary cascade tree. Results similar to those derived here hold for finite trees of any type. In particular, the explicit solution of the Chapman-Kolmogorov equation that is used, Eq. (15), holds for any prime tree.

For a binary cascade tree (Fig. 1) we use a dyadic labeling for the possible states at each level. The *state space* V_n at level l_n will be the set of all products $M^{(1)} \cdots M^{(n)}$ with $M^{(1)} \in M_0, M_1$. There are 2^n elements in V_n and the probability of the state α is

$$\rho_\alpha = P(M^{(1)} \cdots M^{(n)}) = C^{n_0(\alpha)}(1-C)^{n_1(\alpha)} \ , \tag{10}$$

where $n_0(\alpha)$ and $n_1(\alpha)$ are the number of zeros and ones in the dyadic labeling of the state.

The multifractal multiplier models make statements about these probabilities which, as we know, are sufficient to interpret the experimental results on velocity increments over a distance l and the turbulent energy dissipation on a box of size l. These quantities test the invariant probability measure but provide no detailed test of the dynamics of the cascade. If one wants, for example, the time correlations at a point moving with the free-stream velocity of the fluid, one should explicitly consider models for the dynamics, in state space, at each level n. Many different processes are compatible with the same invariant measure ρ_α. The most unstructured process would correspond to the statement that if at time zero one finds the state α, then the transition probability to the state β, at time t, is proportional to ρ_β. For the turbulent cascade the unstructured process does not seem to be natural because the transition probabilities should somehow be related to the lifetime of the eddies, not simply to the stationary measure. If, for example, the lifetime of eddies in the inertial range scales as $l_n/\delta v_n$, we expect large eddies to live longer than small eddies. That is, if at time zero the fluctuation $\delta v_n(x(0))$ at the point $x(0)$ is receiving its energy through a fragmentation history leading to the state α then, a short time $\Delta\tau$ thereafter, we expect to find, at the Lagrangian coordinate $x(\Delta\tau)$, a different state which is nearby in the sense of the natural ultrametric distance in the tree. Recall that the natural distance of two points in a tree is the (ultrametric) distance associated with the p-adic labeling of the tree or, equivalently, the number of steps one has to go back to find the first common ancestor of the two points.

We assume that the dynamics of the turbulent cascade may be described by a Markov process. This amounts to assuming that the set of variables used to characterize the cascade at all length scales at any given time provides a complete description of the dynamical state. That is, no additional set of time-shifted variables is needed to predict the future behavior of the system. To characterize a Markov stochastic process on a tree one has to solve the Chapman-Kolmogorov equation for the transition probabilities,

$$\partial_t p(\gamma t|\beta 0)=\int d\alpha\{W(\gamma|\alpha t)p(\alpha t|\beta 0) - W(\alpha|\gamma t)p(\gamma t|\beta 0)\} \ . \qquad (11)$$

$p(\gamma t|\beta 0)$ denotes the conditional probability to find the state γ at time t given that the state at time zero is β. $W(\gamma|\alpha t)$ is the transition kernel between states α and γ in time t.

For transition kernels that depend only on the distance $W(\gamma|\alpha t)=W(|\gamma-\alpha|)$, Ogielski and Stein [19] found the solution of Eq. (11). Albeverio and Karwowski [20,21] have also constructed the stochastic processes on arbitrary p-adic fields Q_p for the case where the jumping kernels depend only on the distance between p-adic balls (see also [22]). However, it is easy to see from the equation for the probability densities

$$\partial_t\rho(\gamma,t)=\int d\alpha\{W(\gamma|\alpha t)\rho(\alpha,t)-W(\alpha|\gamma t)\rho(\gamma,t)\} \qquad (12)$$

that if $W(\gamma|\alpha t)=W(|\gamma-\alpha|)$ then the invariant density is $\rho(\gamma)=\text{const}$. For the stochastic process of the turbulent cascade we may, in general, require a nonconstant invariant density. From (12) it follows that with

$$W(\gamma|\alpha t)=\rho(\gamma)u(|\gamma-\alpha|) \qquad (13)$$

the invariant density is $\rho(\gamma)$ and, at the same time, full account is taken of the dependence of the transition probability on the distance between the points γ and α in state space. A stochastic process with asymmetric jumping kernels, as in Eq. (13), may therefore be appropriate to describe the dynamics of the turbulent cascade. A general treatment of asymmetric stochastic processes of this type has been given in Ref. [23] for p-adic fields (and adeles). For finite trees the results to be used are those that refer to processes on p-adic balls, which we summarize below (adapted to a finite tree notation).

We denote by $\rho_n^{(\alpha)}$ the asymptotic occupation probability of the state α at the level n of the tree. Then $\rho_{n-r}^{(\alpha)}$ is the occupation probability of the branch that starts at the level $n-r$ and contains α (see Fig. 1). The occupation probabilities at different levels are, of course, related by probability conservation. For example, $\rho_{n-1}^{(3)}=\rho_n^{(3)}+\rho_n^{(4)}$. Also $\rho_{n-1}^{(3)}\equiv\rho_{n-1}^{(4)}$. For the distance-dependent part of the transition kernel in (13) we use the notation $u(n,k)$ for a transition, at the level n, to a distance k (Fig. 2). We then define the quantities

$$\omega_{n,j}^{(\alpha)}=-\sum_{k=j}^{n}[u(n,k)-u(n,k+1)]\rho_{n-k}^{(\alpha)} \qquad (14)$$

with $u(n,n+1)=0$ by definition. Physically we think of the probabilities $\rho_n^{(\alpha)}$ as referring to physical variables at the scale $l_n=l_0 2^{-n}$, l_0 being a typical macroscopic length of the system.

The transition probability, in time t, between the states α and β, solution of the Chapmann-Kolmogorov equation, is [23]

$$p(\beta t|\alpha 0)=\rho_n^{(\beta)}\left\{1+\sum_{k=0}^{n-d_{\alpha\beta}-1}\left[\frac{1}{\rho_{n-(d_{\alpha\beta}+k)}^{(\alpha)}}-\frac{1}{\rho_{n-(d_{\alpha\beta}+k+1)}^{(\alpha)}}\right]e^{t\omega_{n,d_{\alpha\beta}+k+1}^{(\alpha)}}-\frac{1}{\rho_{n-d_{\alpha\beta}}^{(\alpha)}}e^{t\omega_{n,d_{\alpha\beta}}^{(\alpha)}}(1-\delta_{\beta,\alpha})\right\}, \qquad (15)$$

where $d_{\alpha\beta}$ is the tree distance between α and $\beta(d_{\alpha\beta}=|\beta-\alpha|)$.

Once the transition probability is known, the time correlation of a random function f is obtained from

$$\langle f(t)f(0)\rangle=\int d\beta\, d\alpha\, f(\beta)p(\beta t|\alpha 0)f(\alpha)\rho(\alpha) \ .$$

Using Eqs. (14) and (15) one obtains

$$\langle f(t)f(0)\rangle-\langle f\rangle^2=\sum_{\alpha,\beta}\rho_n^{(\alpha)}\rho_n^{(\beta)}f(\alpha)f(\beta)\left\{\sum_{k=0}^{n-d_{\alpha\beta}-1}\left[\frac{1}{\rho^{(\alpha)}_{n-(d_{\alpha\beta}+k)}}-\frac{1}{\rho^{(\alpha)}_{n-(d_{\alpha\beta}+k+1)}}\right]\times\exp\left\{-t\sum_{r=d_{\alpha\beta}+k+1}^{n}[u(n,r)-u(n,r+1)]\rho^{(\alpha)}_{n-r}\right\}-\frac{1}{\rho^{(\alpha)}_{n-d_{\alpha\beta}}}\exp\left\{-t\sum_{r=d_{\alpha\beta}}^{n}[u(n,r)-u(n,r+1)]\rho^{(\alpha)}_{n-r}\right\}(1-\delta_{\beta,\alpha})\right\}. \tag{16}$$

Equation (16) is a very general expression in the sense that it depends only on the form (13) assumed for the transition kernels in the tree. Therefore it may be used as a tool to test any specific hypothesis on the physical nature of the dynamics of the turbulent cascade. As an example, we will compute the time correlations under the assumption that, at level n, transitions involving a jump to a distance k are controlled by the lifetime of the eddies at level $n-k$, that is,

$$u(n,k)=u(n-k)=\left[\frac{2^{n-k}}{A}\right]^{\chi}. \tag{17}$$

A is a normalization factor and χ a scaling exponent for the lifetime of the eddies. Furthermore, for simplicity and to obtain closed analytic expressions, we consider the case $C=\frac{1}{2}$ in Eq. (10). Actually this leads to a uniform probability distribution and differs, for example, from the value $\frac{7}{8}$ chosen in the multifractal β model to fit the data. However, we believe that the qualitative features, namely, the deviation from the exponential decay, are not much affected by our simplifying choice. In any case, accurate results for any C and any set of transition kernels may always be obtained by performing numerically the sums in Eq. (16).

We consider the correlations in the state space V_n, that is, $\langle \alpha(t)\alpha(0)\rangle_n-\langle\alpha\rangle_n^2$, where $\alpha(t)$ takes values on the set of all products $M^{(1)}\cdots M^{(n)}$ with $M^{(i)}\in\{M_0,M_1\}$. Using Eq. (10) ($C=\frac{1}{2}$) and Eq. (17) one obtains

$$\langle \alpha(t)\alpha(0)\rangle_n-\langle\alpha\rangle_n^2=2^{2n}(M_0^2+M_1^2)^n\left\{\sum_{j=1}^{n}2M_0M_1(M_0+M_1)^{2j-2}(M_0^2+M_1^2)^{-j}S_j+S_0\right\}, \tag{18}$$

where for $j\neq 0$

$$S_j=e^{tk_12^{-k_2}}\left\{\sum_{r=j}^{n-1}2^{n-r-1}e^{-tk_12^{(n-r-1)k_2}}-2^{n-j}e^{-tk_12^{(n-j)k_2}}\right\}, \tag{19}$$

and for $j=0$.

$$S_0=e^{tk_12^{-k_2}}\sum_{r=0}^{n-1}2^{n-r-1}e^{-tk_12^{(n-r-1)k_2}} \tag{20}$$

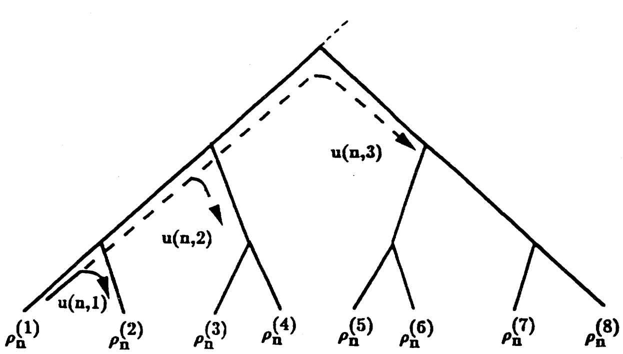

FIG. 2. Stochastic transitions associated with three different tree distances.

with

$$k_1=A^{-\chi}\frac{1-2^{-\chi}}{1-2^{-(\chi+1)}}, \tag{21}$$

$$k_2=\chi+1. \tag{22}$$

Performing the sums in Eq. (18),

$$\langle \alpha(t)\alpha(0)\rangle_n-\langle\alpha\rangle_n^2=2^{2n}(M_0+M_1)^{2n-2}(M_0-M_1)^2\times e^{tk_12^{-k_2}}\sum_{r=1}^{n}\left[\frac{2(M_0^2+M_1^2)}{(M_0+M_1)^2}\right]^{n-r}\times e^{-tk_12^{(n-r)k_2}}. \tag{23}$$

For large n, the sum in Eq. (23) may be approximated by an integral and one obtains finally

$$\langle \alpha(t)\alpha(0)\rangle_n-\langle\alpha\rangle_n^2=2^{2n}(M_0+M_1)^{2n-2}(M_0-M_1)^2\times\frac{e^{tk_12^{-k_2}}}{k_2\log 2}\left[\frac{1}{k_1t}\right]^{\log b/k_2\log 2}\times\left\{\Gamma\left[\frac{\log b}{k_2\log 2},tk_1\right]-\Gamma\left[\frac{\log b}{k_2\log 2},tk_12^{(n-1)k_2}\right]\right\} \tag{24}$$

where

$$b=\frac{2(M_0^2+M_1^2)}{(M_0+M_1)^2} \qquad (25)$$

and $\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}dt$ is the incomplete gamma function.

With the assumption (17), the rigorous result in Eqs. (24) and (25) shows how the hierarchical structure of the cascade tree and the coexistence of many different time scales imply a deviation from the simple exponential behavior, a feature known to occur in other systems as well [24–26].

The time decay implied by Eqs. (24) and (25) may, in principle, be compared with experimentally measured time decays of correlations to obtain information on the scaling exponent χ. Of particular interest are the short and the long time behaviors of Eq. (24) which are, respectively,

$$\left[\frac{1}{t}\right]^{\log b/(\chi+1)\log 2} \quad \text{and} \quad \frac{1}{t}e^{-t[(1-2^{\chi})/A^{\chi}]} .$$

Notice that the correlations we have been computing here are the correlations of the state variable α_n at level n, which equals the product of the multipliers M_i up to level n. These states variables are directly related to the physical variables δv_{l_n} and ϵ_{l_n}, as discussed in the Introduction.

III. CONCLUSION

As we have emphasized before, the stochastic process that we propose for the turbulent cascade is sufficiently general to allow for the test of a large range of different dynamical hypothesis. This would correspond to different choices of the jumping kernels, of which an example was presented in Eq. (17).

The characteristic prediction of the dynamical hypothesis is the shape of the time correlations. Notice that here we are concerned with the time fluctuations of the turbulent cascade itself, not with the changes induced by the overall motion of the fluid. This means that for a fluid in motion with free-stream velocity $\vec{U}$ the correlations to measure, for any cascade observable Δ, are $\langle \Delta(\vec{x}+\vec{U}t,t)\Delta(\vec{x},0)\rangle$. Therefore, for a moving fluid, the experiments to be performed should probe the velocity fluctuations at, at least, two different points, and the separation of the points must also be adjustable. This does not seem feasible in the usual hot wire setting; however, by probing the fluid with two laser beams, the required correlations may, in principle, be extracted.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941); **31**, 338 (1941); **31**, 99 (1941); **32**, 16 (1941).
[2] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[3] F. Anselmet, Y. Cagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[4] A. Vincent and M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991).
[5] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
[6] Z.-S. She, S. Chen, G. Doolen, R. Kraichnan, and S. A. Orszag, Phys. Rev. Lett. **70**, 3251 (1993).
[7] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[8] U. Frisch, P. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).
[9] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics,* edited by M. Gil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), pp. 84–88.
[10] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984).
[11] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).
[12] B. Dubrulle, Phys. Rev. Lett. **73**, 959 (1994).
[13] Z.-S. She and E. C. Waymire, Phys. Rev. Lett. **74**, 262 (1995).
[14] K. R. Sreenivasan and C. Stolovitzky, J. Stat. Phys. **78**, 311 (1995).
[15] J. O'Neil and C. Meneveau, Phys. Fluids A **5**, 158 (1993).
[16] Z.-S. She, E. Jackson, and S. A. Orszag, Proc. R. Soc. London A **434**, 101 (1991).
[17] S. Grossmann and D. Lohse, Phys. Fluids **6**, 611 (1994).
[18] C. Meneveau and T. S. Lund, Phys. Fluids **6**, 2820 (1994).
[19] A. T. Ogielski and D. L. Stein, Phys. Rev. Lett. **55**, 1634 (1985).
[20] S. Albeverio and W. Karwowski, in *Gaussian Random Fields, Proceedings of the 3rd Nagoya Lévy Seminar,* edited by K. Ito and T. Hida (World Scientific, Singapore, 1991), pp. 86–99.
[21] S. Albeverio and W. Karwowski, Stoch. Proc. Theor. Appl. **53**, 1 (1994).
[22] L. Brekke and M. Olson, University of Chicago Report No. EFI-89-23, 1989 (unpublished).
[23] W. Karwowski and R. Vilela Mendes, J. Math. Phys. **35**, 4637 (1994).
[24] R. Kohlrausch, Ann. Phys. (Leipzig) **12**, 393 (1847).
[25] G. Williams and D. C. Watts, Trans. Faraday Soc. **66**, 80 (1970).
[26] T. Palmer, D. Stein, E. Abrahams, and P. W. Anderson, Phys. Rev. Lett. **53**, 958 (1984).